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LETTER TO THE EDITOR

Casimir operators of semidirect sum Lie algebras

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Abstract. A new method for constructing the Casimir operators of a class of non-semisimple Lie algebras is presented. This class includes the semidirect sums of the unitary, orthogonal or symplectic algebras with a Heisenberg-Weyl algebra $\mathrm{w}(n)$. The construction procedure is illustrated with the simple example of the $\mathrm{wu}(n)=\mathrm{w}(n)\oplus\mathrm{u}(n)$ algebras. Its power is then demonstrated on the more difficult case of the $\mathrm{wsp}(2n, R)=\mathrm{w}(n)\oplus\mathrm{sp}(2n, R)$ algebras. Detailed results are given for the eigenvalues of the $\mathrm{wsp}(6, R)$ Casimir operators, relevant to a microscopic theory of nuclear collective motions.

Invariant operators of Lie algebras (also called Casimir operators) are interesting from both mathematical and physical viewpoints. Their eigenvalues can indeed be used to label the irreducible representations (irreps) of the algebras and can also often be identified with the quantum numbers of some physical observables. For the unitary, orthogonal and symplectic algebras, the problems of constructing the Casimir operators and of computing their eigenvalues have been completely solved (Perelomov and Popov 1966, Popov and Perelomov 1967, Popov 1976, Nwachuku and Rashid 1976, 1977, Nwachuku 1979, and references therein). On the other hand, for non-semisimple Lie algebras, such problems remain largely unsolved.

In the present letter, we will show that these important questions can be easily answered for a whole class of non-semisimple Lie algebras. The latter consists of the semidirect sums of the unitary, orthogonal or symplectic algebras with a Heisenberg-Weyl algebra $\mathrm{w}(n)$, i.e. an algebra spanned by n pairs of boson creation and annihilation operators and the unit operator I.

Let us illustrate our Casimir operator construction method with the simple case of the $\mathrm{wu}(n)=\mathrm{w}(n)\oplus\mathrm{u}(n)$ algebras. They are spanned by the operators $E_{ij}=(E_{ji})^{\dagger}$, $B_i^{\dagger}$, $B_i=(B_i^{\dagger})^{\dagger}$, $i, j=1, \ldots, n$, and I, whose non-zero commutators are given by

$$[E_{ij}, E_{kl}]=\delta_{jk}E_{il}-\delta_{il}E_{kj} \tag{1a}$$

$$[B_i, B_j^{\dagger}]=\delta_{ij}I \tag{1b}$$

$$[E_{ij}, B_k^{\dagger}]=\delta_{jk}B_i^{\dagger} \qquad [E_{ij}, B_k]=-\delta_{ik}B_j. \tag{1c}$$

Here E_{ij}, $i, j=1, \ldots, n$, are the $\mathrm{u}(n)$ subalgebra generators, while $B_i^{\dagger}$, B_i, $i=1, \ldots, n$, span, together with I, the invariant Heisenberg-Weyl subalgebra $\mathrm{w}(n)$.

From (1), it follows that the operators $E'_{ij}=(E'_{ji})^{\dagger}$, $i, j=1, \ldots, n$, defined by

$$E'_{ij}=E_{ij}-B_i^{\dagger}B_j \tag{2}$$

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are invariant under w(n), and transform under u(n) in the same way as the generators, i.e.

$$[E'_{ij}, B_k^\dagger] = [E'_{ij}, B_k] = 0 \qquad [E_{ij}, E'_{kl}] = \delta_{jk}E'_{il} - \delta_{il}E'_{kj}. \tag{3}$$

Hence they satisfy commutation relations similar to equation ($1a$), and generate another u(n) algebra, that we shall denote by u'(n) to distinguish it from the u(n) algebra spanned by E_{ij}.

The u'(n) algebra has n independent Casimir operators G'_p, $p = 1, \ldots, n$, given by (Perelomov and Popov 1966)

$$G'_p = \sum_{i_1 i_2 \ldots i_p} E'_{i_1 i_2} E'_{i_2 i_3} \ldots E'_{i_p i_1}. \tag{4}$$

From (3), it is obvious that such operators are also invariant under wu(n):

$$[G'_p, E_{ij}] = [G'_p, B_i^\dagger] = [G'_p, B_i] = 0. \tag{5}$$

Hence, by introducing (2) into (4), we obtain the required wu(n) Casimir operators

$$\hat{G}_{2p} = \sum_{i_1 i_2 \ldots i_p} (E_{i_1 i_2} - B_{i_1}^\dagger B_{i_2})(E_{i_2 i_3} - B_{i_2}^\dagger B_{i_3}) \ldots (E_{i_p i_1} - B_{i_p}^\dagger B_{i_1}) \qquad p = 1, \ldots n \tag{6}$$

which are inhomogeneous polynomials of degree $2p$ in the wu(n) generators.

The same kind of construction also applies to other semidirect sum Lie algebras $wg = \mathrm{w}(n) \oplus g$, where g may be any semisimple, classical algebra. In every case, in the enveloping algebra of wg, one builds a semisimple algebra g' of the same type as g, such that its generators commute with the w(n) ones. Then the Casimir operators of g' are also invariant under wg and may be taken as Casimir operators of the latter. Such a procedure works for both compact and non-compact algebras.

To illustrate the latter point and to show how the knowledge of the g' Casimir operator eigenvalues can be used to compute those of the wg Casimir operators, let us consider the example of the wsp($2n$, R) = w(n) $\oplus$ sp($2n$, R) algebras. Such a case is physically relevant since, as recently shown, the wsp(6, R) algebra finds an interesting application to the microscopic theory of nuclear collective motions (Quesne 1987, 1988).

Let us denote the sp($2n$, R) and w(n) generations by $D_{ij}^\dagger$, $D_{ij} = (D_{ij}^\dagger)^\dagger$, $E_{ij} = (E_{ji})^\dagger$, $i, j = 1, \ldots, n$, and $B_i^\dagger$, B_i, $i = 1, \ldots, n$, respectively. Here E_{ij} are the generators of the u(n) subalgebra, $D_{ij}^\dagger(D_{ij})$ are the sp($2n$, R) raising (lowering) non-compact generators, while $B_i^\dagger$, B_i satisfy equation ($1b$) and are vector operators with respect to sp($2n$, R). The wsp($2n$, R) irreps, relevant to the theory of nuclear collective motions, are positive discrete series ones, characterised by their lowest weight and denoted by $\langle\langle\mathbf{\Omega}\rangle\rangle$, where $\mathbf{\Omega} = \Omega_1\Omega_2 \ldots \Omega_n$ (Quesne 1987, 1988). Their lowest weight state (LWS) $|\mathbf{\Omega}\rangle$ satisfies the following equations:

$$\begin{aligned} &E_{ii}|\mathbf{\Omega}\rangle = \Omega_{n+1-i}|\mathbf{\Omega}\rangle \qquad E_{ij}|\mathbf{\Omega}\rangle = 0 \qquad (i > j) \\ &D_{ij}|\mathbf{\Omega}\rangle = 0 \qquad B_i|\mathbf{\Omega}\rangle = 0. \end{aligned} \tag{7}$$

For the wsp($2n$, R) generators, it is convenient to use the more compact notation $\Lambda_{\alpha\beta}$, V_α, where greek indices run over $\pm 1, \ldots, \pm n$, and

$$\begin{aligned} &\Lambda_{ij} = D_{ij}^\dagger \qquad \Lambda_{-i,-j} = D_{ij} \qquad \Lambda_{i,-j} = \Lambda_{-j,i} = E_{ij} \\ &V_i = B_i^\dagger \qquad V_{-i} = B_i \qquad i, j = 1, \ldots, n. \end{aligned} \tag{8}$$

These operators satisfy the following symmetry and Hermiticity properties:

$$\Lambda_{\alpha\beta} = \Lambda_{\beta\alpha} = (\Lambda_{-\alpha,-\beta})^\dagger \tag{9a}$$

$$V_\alpha = (V_{-\alpha})^\dagger \tag{9b}$$

and their commutators are given by

$$[\Lambda_{\alpha\beta}, \Lambda_{\gamma\delta}] = g_{\gamma\alpha}\Lambda_{\beta\delta} + g_{\gamma\beta}\Lambda_{\alpha\delta} + g_{\delta\beta}\Lambda_{\gamma\alpha} + g_{\delta\alpha}\Lambda_{\gamma\beta} \tag{10a}$$

$$[V_\alpha, V_\beta] = g_{\beta\alpha} I \qquad [\Lambda_{\alpha\beta}, V_\alpha] = g_{\gamma\alpha} V_\beta + g_{\gamma\beta} V_\alpha \tag{10b}$$

in terms of the metric tensor

$$g_{\alpha\beta} = (\alpha/|\alpha|)\delta_{\alpha,-\beta}. \tag{11}$$

The operators $\Lambda'_{\alpha\beta}$, defined by

$$\Lambda'_{\alpha\beta} = \Lambda_{\alpha\beta} - \tfrac{1}{2}(V_\alpha V_\beta + V_\beta V_\alpha) = \Lambda_{\alpha\beta} - V_\alpha V_\beta - \tfrac{1}{2} g_{\alpha\beta} I \tag{12}$$

satisfy equations similar to $(9a)$ and $(10a)$, and therefore span an sp'$(2n, R)$ algebra. Since, in addition, they commute with the w(n) generators V_α, the sp'$(2n, R)$ Casimir operators, given by (Nwachuku and Rashid 1976, 1977, Nwachuku 1979)

$$\begin{aligned} G'_{2p} &= \tfrac{1}{2}\operatorname{tr}(\Lambda')^{2p} \\ &= \tfrac{1}{2} \sum_{\alpha_1\alpha_2\ldots\alpha_{2p}} \sum_{\beta_1\beta_2\ldots\beta_{2p}} g^{\alpha_1\beta_1} g^{\alpha_2\beta_2} \ldots g^{\alpha_{2p}\beta_{2p}} \Lambda'_{\beta_1\alpha_2}\Lambda'_{\beta_2\alpha_3}\ldots\Lambda'_{\beta_{2p}\alpha_1} \\ & p = 1, \ldots, n \end{aligned} \tag{13}$$

are invariant under wsp$(2n, R)$.

At first sight, these operators seem to be of degree $4p$ in the wsp$(2n, R)$ generators. However, from $(10b)$, it follows that

$$\operatorname{tr} V^2 = \sum_{\alpha\beta} g^{\alpha\beta} V_\beta V_\alpha = -nI. \tag{14}$$

Hence the highest degree terms in G'_{2p} are proportional to $\operatorname{tr}[(\Lambda \cdot VV)^p + (VV \cdot \Lambda)^p]$, so that G'_{2p} is only of degree $3p$ in the wsp$(2n, R)$ generators. Since, moreover, G'_{2p} contains a term $\gamma_p I$, proportional to the unit operator, let us define the wsp$(2n, R)$ Casimir operators $\hat{G}_{3p}$ by the relation

$$\hat{G}_{3p} = G'_{2p} - \gamma_p I \qquad p = 1, \ldots, n. \tag{15}$$

The value of γ_p, $p = 1, \ldots, n$, will be determined below.

The eigenvalues $\hat{g}_{3p}(\mathbf{\Omega})$ of $\hat{G}_{3p}$, corresponding to $\langle\langle\mathbf{\Omega}\rangle\rangle$, can be calculated by acting with $\hat{G}_{3p}$ on the irrep LWS$|\mathbf{\Omega}\rangle$. For such a purpose, we note that the sp'$(2n, R)$ generators can be rewritten as

$$\begin{aligned} D'^{\dagger}_{ij} &= \Lambda'_{ij} = D^{\dagger}_{ij} - B^{\dagger}_i B^{\dagger}_j \qquad D'_{ij} = \Lambda'_{-i,-j} = D_{ij} - B_i B_j \\ E'_{ij} &= \Lambda'_{i,-j} = E_{ij} - B^{\dagger}_i B_j - \tfrac{1}{2}\delta_{ij} I. \end{aligned} \tag{16}$$

Hence the wsp$(2n, R)$ LWS $|\mathbf{\Omega}\rangle$ satisfies the relations

$$E'_{ii}|\mathbf{\Omega}\rangle = \Omega'_{n+1-i}|\mathbf{\Omega}\rangle \qquad E'_{ij}|\mathbf{\Omega}\rangle = 0 (i > j) \qquad D'_{ij}|\mathbf{\Omega}\rangle = 0 \tag{17}$$

where

$$\Omega'_i = \Omega_i - \tfrac{1}{2}. \tag{18}$$

It is therefore the LWS of an sp'$(2n, R)$ irrep $\langle\mathbf{\Omega}'\rangle$, where $\mathbf{\Omega}' = \Omega'_1\Omega'_2\ldots\Omega'_n$. The corresponding eigenvalue equation for G'_{2p} is

$$G'_{2p}|\mathbf{\Omega}\rangle = g_{2p}(\mathbf{\Omega}')|\mathbf{\Omega}\rangle \tag{19}$$

where the explicit expression of $g_{2p}(\mathbf{\Omega}')$ in terms of $\mathbf{\Omega}'$ can be found from Nwachuku's tables (1979).

After introducing (18) into $g_{2p}(\boldsymbol{\Omega}')$, the latter separates into an $\boldsymbol{\Omega}$-dependent term and a constant, which are the eigenvalue $\hat{g}_{3p}(\boldsymbol{\Omega})$ of $\hat{G}_{3p}$, and the constant γ_p, respectively. In the wsp(6, R) case, for instance, the results are

$$\hat{g}_3(\boldsymbol{\Omega}) = \sum_i \Omega_i(\Omega_i - 2i - 1) \qquad \hat{g}_6(\boldsymbol{\Omega}) = \sum_i \Omega_i(\Omega_i - 2i - 1)[\Omega_i(\Omega_i - 2i - 1) + \tfrac{9}{2} + 2i + 2i^2]$$

$$\begin{aligned}\hat{g}_g(\boldsymbol{\Omega}) = \sum_i \Omega_i(\Omega_i - 2i - 1) &\\ \times\Big(\Omega_i^2(\Omega_i - 2i - 1)^2 &+ (\tfrac{223}{4} + 3i - 3i^2)\Omega_i(\Omega_i - 2i - 1) \\ &- \tfrac{14\,053}{16} + \tfrac{281}{2}i + \tfrac{287}{2}i^2 + 6i^3 + 3i^4 - \frac{29}{2}\sum_{j\neq i}\Omega_j(\Omega_j - 2j - 1)\Big)\end{aligned} \tag{20}$$

and

$$\gamma_1 = \tfrac{27}{4} \qquad \gamma_2 = \tfrac{1971}{16} \qquad \gamma_3 = \tfrac{159\,435}{64}. \tag{21}$$

By using equation (16), expressions of the Casimir operators $\hat{G}_{3p}$ in terms of the original generators $D_{ij}^\dagger$, D_{ij}, E_{ij}, $B_i^\dagger$ and B_i can also be obtained. The cubic Casimir operator, for instance, can be rewritten as

$$\hat{G}_3 = \sum_{ij} (D_{ij}^\dagger B_i B_j + B_i^\dagger B_j^\dagger D_{ij} - 2B_i^\dagger E_{ji} B_j) + \sum_i (2B_i^\dagger B_i - E_{ii}) + G_2 \tag{22}$$

where

$$G_2 = \sum_{ij} [E_{ij}E_{ji} - D_{ij}^\dagger D_{ij}] - (n+1)\sum_i E_{ii} \tag{23}$$

is the sp($2n$, R) quadratic Casimir operator. Up to a multiple of the unit operator, equation (22) generalises to wsp($2n$, R), a result previously obtained by Hughes (1981) for wsp(2, R).

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